

$\mathcal{N} = 4$ chiral supermultiplet interacting with a magnetic field

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Abstract

We couple $\mathcal{N}=4$ chiral supermultiplet with an auxiliary $\mathcal{N}=4$ fermionic supermultiplet containing on-shell four physical fermions and four auxiliary bosons. The latter ones play the role of isospin variables. We choose the very specific coupling which results in a component action containing only time derivatives of fermionic components presented in the auxiliary supermultiplet, which therefore may be dualized into auxiliary ones. The resulting component action describes the interaction of the chiral supermultiplet with a magnetic field constant on the pseudo-sphere $SU(1,1)/U(1)$. Then we specify the prepotential of our theory to get, in the bosonic sector, the action for the particle moving over the pseudo-sphere – Lobachevsky space. We provided also the Hamiltonian formulation of this system and show that the full symmetry group of our system is $SU(1,1) \times U(1)$. The currents forming the $su(1,1)$ algebra are modified, as compared to the bosonic case, by the fermionic and isospin terms, while the additional $u(1)$ current contains only isospin variables.

One of the most important features of our construction is the presence in the Hamiltonian and supercharges of all currents of the isospin group $SU(2)$. Despite the fact that two of the $su(2)$ currents $\{T, \bar{T}\}$ enter the Hamiltonian only through the Casimir operator of the $SU(2)$ group, they cannot be dropped out, even after fixing the total isospin of the system, because these currents themselves enter into the supercharges.

We also present the Hamiltonian and supercharges describing the motion of a particle over the sphere S^2 in the background of constant magnetic field. In this case the additional isospin currents form the $su(1,1)$ algebra.

1 Introduction

It is a well known fact that the Hamiltonian formalism provides the most natural framework for the description of quantum mechanics of a isospin-carrying particle in the background of magnetic monopole [1]. However, this approach meets a lot of problems when we attempt to add supersymmetry, especially $\mathcal{N}=4$ supersymmetry. A well known example of such difficulties appears when we try to find the $\mathcal{N}=4$ supersymmetric extension of the particle moving over the pseudo-sphere (Lobachevsky space) or the sphere S^2 in the field of a Dirac monopole. Indeed, the simplest analysis shows that the corresponding bosonic Hamiltonian

$$\mathcal{H}_{bos} = (1 - x\bar{x})^2 \left(p + \frac{i\bar{x}}{1 - x\bar{x}} J \right) \left(\bar{p} - \frac{ix}{1 - x\bar{x}} J \right) \quad (1.1)$$

cannot be directly $\mathcal{N}=4$ supersymmetrized within the most natural chiral $\mathcal{N}=4$ supermultiplet (see e.g. [2]). As we will show in the present paper, in order to construct the $\mathcal{N}=4$ version of the Hamiltonian (1.1) one should include into the game, together with J , all isospin currents spanning $su(2)$ or $su(1,1)$ algebras in the cases of Lobachevsky space or sphere S^2 , respectively.

Clearly, the Lagrangian description in terms of superfields is the most suitable for supersymmetric theories. Therefore, to enjoy all power of the superspace approach one has to use the Lagrangian formalism. Switching to the Lagrangian description immediately raised the question: which supermultiplet do the isospin degrees of freedom belong to? Indeed, working in the Lagrangian formalism we have to treat the isospin vector J as a composite one. Thus, the corresponding isospin degrees of freedom have to enter the Lagrangian with a kinetic term of the first-order in time derivatives – only in this case it will be possible to construct the isospin current as bilinear combinations of isospin variables after quantization. A first solution of this problem has been proposed in [3], where auxiliary bosonic and fermionic degrees of freedom constitute an auxiliary gauge supermultiplet. Then this idea has been used for the construction of $\mathcal{N}=4$ superconformal mechanics with isospin degrees of freedom [4]. The basic framework for this approach is Harmonic superspace adapted to one dimension [5].

Another approach [6], which we will exploit here, uses the ordinary superspace and ordinary superfields. Its key feature is just a specific coupling between two $\mathcal{N}=4$ supermultiplets involved in the game, which gives a kinetic term of the first order in time derivatives for the bosonic isospin variables. In Section 2, following these arguments, we construct a coupling of $\mathcal{N}=4$ chiral supermultiplet and auxiliary $\mathcal{N}=4$ fermionic supermultiplet of this type. The resulting component action describes the interaction of the chiral supermultiplet with a magnetic field which is constant on the pseudo-sphere $SU(1,1)/U(1)$. In Section 3 we specify the prepotential of our theory to get in the bosonic sector the action for the particle moving over the pseudo-sphere – Lobachevsky space. In Section 4 we provide the Hamiltonian formulation of this system explicitly demonstrating that the constructed system describes the $\mathcal{N}=4$ supersymmetric isospin carrying particle moving in the Lobachevsky space and interacting with the constant magnetic field. Our action possesses the $SU(1,1)$ invariance, similarly to its bosonic analog. In Section 5 we consider some related problems and extensions. Finally, we conclude with a summary of results and some possible further directions of study.

2 Superspace action

The appropriate way to construct supersymmetric theories is to use a superspace approach in which the supersymmetry is manifest. In $\mathcal{N}=4$ superspace $\mathbf{R}^{(1|4)} = (t, \theta_i, \bar{\theta}^i)$, where the spinor derivatives satisfy the standard anticommutation relations

$$\{D^i, \bar{D}_j\} = 2i\delta_j^i \partial_t, \quad \{D^i, D^j\} = 0, \quad \{\bar{D}_i, \bar{D}_j\} = 0, \quad (2.1)$$

the chiral supermultiplet is described by a complex scalar superfield Z subjected to the constraints

$$D^i Z = 0, \quad \bar{D}_i \bar{Z} = 0. \quad (2.2)$$

It contains a complex physical boson $\{z, \bar{z}\}$, a doublet of physical fermions $\{\psi_i, \bar{\psi}^i\}$ and auxiliary fields $\{A, \bar{A}\}$, which can be defined as

$$z = Z|, \quad \bar{z} = \bar{Z}|, \quad \psi_i = (\bar{D}_i Z)|, \quad \bar{\psi}^i = - (D^i \bar{Z})|, \quad A = i (\bar{D}_i \bar{D}^i Z)|, \quad \bar{A} = -i (D^i D_i \bar{Z})|, \quad (2.3)$$

where $|$ in the r.h.s. denotes, as usual, the $\theta = \bar{\theta} = 0$ limit. The most general σ -model type superfield action for the chiral supermultiplet is defined through an arbitrary prepotential $G(Z, \bar{Z})$ as ¹

$$S_{(Z, \bar{Z})} = \int dt d^2\theta d^2\bar{\theta} G(Z, \bar{Z}). \quad (2.4)$$

One of the possibilities to invent semi-dynamical isospin variables in the Lagrangian of some supersymmetric mechanics, and therefore to introduce the interaction with magnetic fields, is to introduce a specific coupling of the basic supermultiplet (i.e. chiral supermultiplet in the case at hands) with another “auxiliary” superfield which contains these isospin variables together with auxiliary fermions. The most essential feature of such a coupling is that it has to be a first-order kinetic term for the isospin variables. The approach, which we will follow in this Letter, has been elaborated in [6, 7, 8, 9]. There, isospin variables and auxiliary fermions were put in a doublet of fermionic superfields $\{\Xi^i, \bar{\Xi}^i\}$ subjected to the irreducible conditions [10]

$$D^{(i} \Xi^{k)} = 0, \quad \bar{D}^{(i} \bar{\Xi}^{k)} = 0, \quad D^{(i} \bar{\Xi}^{k)} = 0, \quad \bar{D}^{(i} \Xi^{k)} = 0, \quad (2.5)$$

The constraints (2.5) leave in the superfields $\{\Xi^i, \bar{\Xi}^i\}$ four fermionic and four bosonic components

$$\xi^i = \Xi^i|, \quad \bar{\xi}_i = \bar{\Xi}_i|, \quad v = (D^i \Xi_i)|, \quad w = (\bar{D}_i \bar{\Xi}^i)|, \quad \bar{w} = -(D^i \bar{\Xi}_i)|, \quad \bar{v} = (\bar{D}_i \bar{\Xi}^i)|. \quad (2.6)$$

Following [6] we introduce the coupling of these chiral superfield Z and auxiliary one Ξ^i as

$$S = S_{(Z, \bar{Z})} + S_{(Z, \Xi)} = \int dt d^2\theta d^2\bar{\theta} \left(G(Z, \bar{Z}) + F(Z, \bar{Z}) \Xi^i \bar{\Xi}_i \right), \quad (2.7)$$

¹We define the integration measure to be $d^2\theta d^2\bar{\theta} = \frac{1}{16} D^i D_i \bar{D}_j \bar{D}^j$.

where $F(Z, \bar{Z})$ is a real, for the time being arbitrary function.

To ensure that in the action (2.7) the fermionic fields ξ^i and $\bar{\xi}_i$ appear only through the time derivatives and therefore one may replace the time derivatives of ξ^i and $\bar{\xi}_i$ by new fermionic fields $\hat{\xi}^i$ and $\hat{\bar{\xi}}_i$ as

$$\hat{\xi}^i = \dot{\xi}^i, \quad \hat{\bar{\xi}}_i = \dot{\bar{\xi}}_i, \quad (2.8)$$

one has to impose some additional constraint on the function $F(Z, \bar{Z})$. One may check that in the present case this constraint is incredibly strong

$$\frac{\partial^2 F(Z, \bar{Z})}{\partial Z \partial \bar{Z}} = 0 \quad \Rightarrow \quad F(Z, \bar{Z}) = f(Z) + \bar{f}(\bar{Z}). \quad (2.9)$$

Moreover, due to the arbitrariness of our prepotential $G(Z, \bar{Z})$, without loss of generality, one may assume, that the function $F(Z, \bar{Z})$ in (2.9) has the form

$$F(Z, \bar{Z}) = Z + \bar{Z}, \quad (2.10)$$

and thereby, the superfield action of our model acquires an extremely simple form

$$S = \int dt d^2\theta d^2\bar{\theta} \left(G(Z, \bar{Z}) + (Z + \bar{Z}) \Xi^i \Xi_i \right). \quad (2.11)$$

After integrating over θ 's in (2.11), replacing the fermionic components (2.8) and excluding the auxiliary fields $\{A, \bar{A}\}$ and $\{\hat{\xi}^i, \hat{\bar{\xi}}_i\}$ by their equations of motion

$$A = -\frac{i}{g} (w\bar{v} - g_z \psi^k \psi_k), \quad \hat{\xi}^i = \frac{i}{4(z + \bar{z})} (v\psi^i - w\bar{\psi}^i), \quad (2.12)$$

we will get the following on-shell action:

$$\begin{aligned} S^{(\text{on-shell})} &= \int dt \left\{ g \dot{z} \dot{\bar{z}} - \frac{i}{4} g \left(\psi_i \dot{\bar{\psi}}^i - \dot{\psi}_i \bar{\psi}^i \right) - \frac{i}{4} \left(g_{\bar{z}} \dot{z} - g_z \dot{\bar{z}} \right) \psi_i \bar{\psi}^i \right. \\ &+ \frac{i}{8} (z + \bar{z}) (\dot{v}\bar{v} - v\dot{\bar{v}} + \dot{w}\bar{w} - w\dot{\bar{w}}) + \frac{i}{8} (v\bar{v} - w\bar{w}) (\dot{z} - \dot{\bar{z}}) \\ &- \frac{1}{8(z + \bar{z})} (v\bar{v} - w\bar{w}) \psi_i \bar{\psi}^i + \frac{1}{16} \left(\frac{2}{z + \bar{z}} + \frac{g_z}{g} \right) v\bar{w} \psi^i \psi_i + \frac{1}{16} \left(\frac{2}{z + \bar{z}} + \frac{g_{\bar{z}}}{g} \right) w\bar{v} \bar{\psi}_i \bar{\psi}^i \\ &\left. - \frac{1}{16g} v\bar{v} w\bar{w} + \frac{1}{16} \left(g_{z\bar{z}} - \frac{g_z g_{\bar{z}}}{g} \right) \psi^i \psi_i \bar{\psi}_k \bar{\psi}^k \right\}, \end{aligned} \quad (2.13)$$

where the metric g is defined as $g = \frac{\partial^2 G(z, \bar{z})}{\partial z \partial \bar{z}}$.

The action (2.13) is the main result of this Letter. It describes the $\mathcal{N}=4$ supersymmetric isospin particle moving in the background of some magnetic field which is specified by the last term in the second line of the action (2.13). Up to now our consideration was quite general, and the prepotential $G(Z, \bar{Z})$, as well as induced metrics in the configuration space g , were arbitrary. Let us now specify the prepotential to get the cases of special interest and to clarify the structure of the interaction.

3 Lobachevsky space: Lagrangian

From the explicit form of our superspace action (2.11) one could see that the $U(1)$ symmetry, which seems to be manifest in any theory based on the $\mathcal{N}=4$ chiral supermultiplet, is realized as

$$\delta Z = ia, \quad \delta \bar{Z} = -ia. \quad (3.1)$$

Therefore, it makes sense to limit ourselves to considering the prepotential compatible with this realization. This means that one should choose

$$G(z, \bar{z}) = G(z + \bar{z}) \quad \Rightarrow \quad g(z, \bar{z}) = g(z + \bar{z}). \quad (3.2)$$

Among the two-dimensional system with the metric (3.2) there is one special case with

$$g(z, \bar{z}) = \frac{1}{(z + \bar{z})^2}. \quad (3.3)$$

This is just the well known Lobachevsky space. The mechanics on such a configuration space is commonly used for testing different properties of the theory (see e.g. [11, 12]). Let us analyze our action (2.13) for this special case in details.

Making the redefinition of isospin variables as

$$\tilde{v} = \frac{\sqrt{z + \bar{z}}}{2} v, \quad \tilde{w} = \frac{\sqrt{z + \bar{z}}}{2} w, \quad \bar{\tilde{v}} = \frac{\sqrt{z + \bar{z}}}{2} \bar{v}, \quad \bar{\tilde{w}} = \frac{\sqrt{z + \bar{z}}}{2} \bar{w}, \quad (3.4)$$

and performing the following change of variables:

$$\begin{aligned} z &= \frac{1+x}{1-\bar{x}}, \quad \bar{z} = \frac{1+\bar{x}}{1-x} \Rightarrow z + \bar{z} = \frac{2(1-x\bar{x})}{(1-x)(1-\bar{x})}, \quad i \frac{\dot{z} - \dot{\bar{z}}}{z + \bar{z}} = -\mathcal{A} + \dot{f}, \\ \text{where } \mathcal{A} &= i \frac{\dot{x}\bar{x} - x\dot{\bar{x}}}{1-x\bar{x}}, \quad f = i \log \left(\frac{1-\bar{x}}{1-x} \right), \\ \text{and } \tilde{\psi}_i &= \frac{(1-x)(1-\bar{x})}{2\sqrt{2}} \psi_i, \quad \bar{\tilde{\psi}}^i = \frac{(1-x)(1-\bar{x})}{2\sqrt{2}} \bar{\psi}^i, \end{aligned} \quad (3.5)$$

one may bring the action (2.13) to the form

$$\begin{aligned} S^{(\text{on-shell})} &= \int dt \left\{ \frac{1}{(1-x\bar{x})^2} \dot{x}\dot{\bar{x}} - \frac{i}{2(1-x\bar{x})^2} (\tilde{\psi}_i \dot{\bar{\tilde{\psi}}}^i - \dot{\tilde{\psi}}_i \bar{\tilde{\psi}}^i) + \frac{1}{(1-x\bar{x})^2} (\mathcal{A} - \dot{f}) \tilde{\psi}_i \bar{\tilde{\psi}}^i \right. \\ &+ \frac{i}{2} (\dot{\tilde{v}}\bar{\tilde{v}} - \tilde{v}\dot{\bar{\tilde{v}}} + \dot{\tilde{w}}\bar{\tilde{w}} - \tilde{w}\dot{\bar{\tilde{w}}}) - \frac{1}{2} (\tilde{v}\bar{\tilde{v}} - \tilde{w}\bar{\tilde{w}}) (\mathcal{A} - \dot{f}) \\ &\left. - \tilde{v}\bar{\tilde{v}}\tilde{w}\bar{\tilde{w}} - \frac{1}{(1-x\bar{x})^2} (\tilde{v}\bar{\tilde{v}} - \tilde{w}\bar{\tilde{w}}) \tilde{\psi}_i \bar{\tilde{\psi}}^i + \frac{1}{2(1-x\bar{x})^4} \tilde{\psi}^i \tilde{\psi}_i \bar{\tilde{\psi}}_k \bar{\tilde{\psi}}^k \right\}. \end{aligned} \quad (3.6)$$

Removing the quantity f from the action (3.6) by the gauge transformation

$$\begin{aligned} \hat{v} &= e^{-\frac{i}{2}f} \tilde{v}, \quad \hat{w} = e^{\frac{i}{2}f} \tilde{w}, \quad \bar{\hat{v}} = e^{-\frac{i}{2}f} \bar{\tilde{v}}, \quad \bar{\hat{w}} = e^{\frac{i}{2}f} \bar{\tilde{w}}, \\ \chi_i &= \frac{e^{if}}{1-x\bar{x}} \tilde{\psi}_i, \quad \bar{\chi}^i = \frac{e^{-if}}{1-x\bar{x}} \bar{\tilde{\psi}}^i, \end{aligned} \quad (3.7)$$

we will finally get the action

$$\begin{aligned} S_{final}^{(\text{on-shell})} &= \int dt \left\{ \frac{1}{(1-x\bar{x})^2} \dot{x}\dot{\bar{x}} - \frac{i}{2} (\chi_i \dot{\bar{\chi}}^i - \dot{\chi}_i \bar{\chi}^i) + \mathcal{A} \chi_i \bar{\chi}^i \right. \\ &+ \frac{i}{2} (\dot{\hat{v}}\bar{\hat{v}} - \hat{v}\dot{\bar{\hat{v}}} + \dot{\hat{w}}\bar{\hat{w}} - \hat{w}\dot{\bar{\hat{w}}}) - \frac{1}{2} (\hat{v}\bar{\hat{v}} - \hat{w}\bar{\hat{w}}) \mathcal{A} \\ &\left. - \hat{v}\bar{\hat{v}}\hat{w}\bar{\hat{w}} - (\hat{v}\bar{\hat{v}} - \hat{w}\bar{\hat{w}}) \chi_i \bar{\chi}^i + (\chi_i \bar{\chi}^i)^2 \right\}. \end{aligned} \quad (3.8)$$

The action (3.8) describes the motion of the particle in the Lobachevsky space in the background of a magnetic field \mathcal{A} with the field strength

$$F = B(x, \bar{x}) dx \wedge d\bar{x} = \frac{2i}{(1-x\bar{x})^2} dx \wedge d\bar{x}. \quad (3.9)$$

Clearly, this is just a constant magnetic field on the pseudo-sphere $SU(1,1)/U(1)$ with the metric $g = \frac{1}{(1-x\bar{x})^2}$.

One should note that the Lobachevsky space, being the coset space $SU(1,1)/U(1)$, possesses $SU(1,1)$ symmetry. In the Lagrangian form this symmetry is hidden. Moreover, the treatment of the variables $\{\hat{v}, \bar{\hat{v}}, \hat{w}, \bar{\hat{w}}\}$ as the isospin ones refers to the Hamiltonian approach, where they will obey proper Dirac brackets. In the next Section we will present the corresponding Hamiltonian and supercharges and analyze the symmetries of our action (3.8) in full details.

4 Lobachevsky space: Hamiltonian and Symmetries

In order to find the classical Hamiltonian, we follow the standard procedure for quantizing a system with bosonic and fermionic degrees of freedom. From the action (3.8) we define the momenta $\{p, \bar{p}, \pi^i, \bar{\pi}_i, p_v, \bar{p}_v, p_w, \bar{p}_w\}$ conjugated to $\{x, \bar{x}, \chi_i, \bar{\chi}^i, \hat{v}, \bar{\hat{v}}, \hat{w}, \bar{\hat{w}}\}$ respectively, as

$$p = \frac{\dot{\bar{x}}}{(1-x\bar{x})^2} + \frac{i\bar{x}}{1-x\bar{x}} \left(\chi_i \bar{\chi}^i - \frac{1}{2}(\hat{v} \bar{\hat{v}} - \hat{w} \bar{\hat{w}}) \right), \quad \pi^i = \frac{i}{2} \dot{\bar{\chi}}^i, \quad p_v = \frac{i}{2} \dot{\bar{\hat{v}}}, \quad p_w = \frac{i}{2} \dot{\bar{\hat{w}}}, \quad (4.1)$$

and introduce Dirac brackets for the canonical variables

$$\{x, p\} = 1, \quad \{\bar{x}, \bar{p}\} = 1, \quad \{\chi_i, \bar{\chi}^k\} = i\delta_i^k, \quad \{\hat{v}, \bar{\hat{v}}\} = -i, \quad \{\hat{w}, \bar{\hat{w}}\} = -i. \quad (4.2)$$

Analyzing the structure of the action (3.8) one may see that the isospin variables enter the action in rather special combinations. That is why it is convenient to introduce the following currents T, \bar{T}, J constructed from the isospin variables

$$T = \hat{v} \bar{\hat{w}}, \quad \bar{T} = \bar{\hat{v}} \hat{w}, \quad J = \frac{1}{2}(\hat{v} \bar{\hat{v}} - \hat{w} \bar{\hat{w}}). \quad (4.3)$$

These currents form the $su(2)$ algebra with respect to the Dirac brackets (4.2)

$$\{T, \bar{T}\} = 2iJ, \quad \{J, T\} = iT, \quad \{J, \bar{T}\} = -i\bar{T}. \quad (4.4)$$

Now one may check that the supercharges

$$\begin{aligned} \mathcal{Q}_i &= (1-x\bar{x})p\chi_i + i\bar{x}J\chi_i + i\bar{T}\bar{\chi}_i - \frac{i}{2}\bar{x}\chi^k\chi_k\bar{\chi}_i, \\ \bar{\mathcal{Q}}^i &= (1-x\bar{x})\bar{p}\bar{\chi}^i - ixJ\bar{\chi}^i + iT\chi^i - \frac{i}{2}x\bar{\chi}_k\bar{\chi}^k\chi^i, \end{aligned} \quad (4.5)$$

and the Hamiltonian

$$\mathcal{H} = (1-x\bar{x})^2 \left(p + \frac{i\bar{x}}{1-x\bar{x}} \left(J - \chi_i \bar{\chi}^i \right) \right) \left(\bar{p} - \frac{ix}{1-x\bar{x}} \left(J - \chi_i \bar{\chi}^i \right) \right) + 2J\chi_i \bar{\chi}^i - (\chi_i \bar{\chi}^i)^2 + T\bar{T}, \quad (4.6)$$

form the $\mathcal{N}=4$ Poincaré superalgebra

$$\{\mathcal{Q}_i, \bar{\mathcal{Q}}^k\} = i\delta_i^k \mathcal{H}. \quad (4.7)$$

From the explicit form of the Hamiltonian one may see that isospin variables enter it only through the $su(2)$ currents $\{T, \bar{T}, J\}$ which commute with everything, excluding themselves. In addition, the Casimir operator $\mathcal{C}_{SU(2)}$ of the $SU(2)$ group given by

$$\mathcal{C}_{SU(2)} = T\bar{T} + J^2, \quad (4.8)$$

commutes with the Hamiltonian and supercharges. Thus, the relations (4.3) define the classical isospin matrices. At the same time the fermions enter the Hamiltonian only through the $u(1)$ current combination $\chi_i \bar{\chi}^i$ thus providing a description for the fermionic spin degrees of freedom.

Therefore, we conclude that the Hamiltonian (4.6) indeed describes the motion of the $\mathcal{N}=4$ supersymmetric isospin carrying particle in the Lobachevsky space in the constant magnetic field.

Now it is time to analyze the symmetries of our Hamiltonian (4.6). In the absence of any interaction the $SU(1,1)$ invariance of the bosonic Hamiltonian is extended to the supersymmetric case by the following currents:

$$R = p - \bar{x}^2 \bar{p} + i\bar{x} \chi_i \bar{\chi}^i, \quad \bar{R} = \bar{p} - x^2 p - ix \chi_i \bar{\chi}^i, \quad U = i(xp - \bar{x}\bar{p}) - \chi_i \bar{\chi}^i. \quad (4.9)$$

These currents span the $su(1,1)$ algebra

$$\{R, \bar{R}\} = -2iU, \quad \{U, R\} = iR, \quad \{U, \bar{R}\} = -i\bar{R}, \quad (4.10)$$

and commute with the supercharges (4.5) and the Hamiltonian (4.6) when $J = T = \bar{T} = 0$.

It is not too hard to check that the slightly modified currents

$$\mathcal{R} = R - i\bar{x}J, \quad \bar{\mathcal{R}} = \bar{R} + ixJ, \quad \mathcal{U} = U + J \quad (4.11)$$

form the same $su(1,1)$ algebra (4.10) and commute now with the full supercharges (4.5) and the Hamiltonian (4.6). Thus, our system, describing the $\mathcal{N}=4$ supersymmetric isospin carrying particle moving in the Lobachevsky space and interacting with the constant magnetic field, possesses the $SU(1,1)$ invariance, similarly to its bosonic analog. Moreover, if we will introduce the Casimir operator $\mathcal{C}_{SU(1,1)}$ of the $SU(1,1)$ group as

$$\mathcal{C}_{SU(1,1)} = \mathcal{R}\overline{\mathcal{R}} - \mathcal{U}^2, \quad (4.12)$$

then one may rewrite our Hamiltonian (4.6) as just a sum of two Casimir operators

$$\mathcal{H} = \mathcal{C}_{SU(1,1)} + \mathcal{C}_{SU(2)}. \quad (4.13)$$

It is clear now that the full symmetry of the Hamiltonian (4.6) is

$$SU(1,1) \times U(1) \propto \{\mathcal{R}, \overline{\mathcal{R}}, \mathcal{U}, J\}. \quad (4.14)$$

One should stress that our $\mathcal{N}=4$ system unavoidably contains all currents of the $su(2)$ algebra $\{T, \overline{T}, J\}$. The last term in the Hamiltonian (4.6) is crucial for $\mathcal{N}=4$ supersymmetry and cannot be dropped out. That is why the standard procedure of $\mathcal{N}=4$ supersymmetrization of the bosonic Hamiltonian describing the motion of the particle over the Lobachevsky space (1.1) fails without introducing additional currents $\{T, \overline{T}\}$.

It is interesting to note that the total isospin of the system can be fixed as

$$\mathcal{C}_{SU(2)} = T\overline{T} + J^2 = M_0 = \text{const.} \quad (4.15)$$

With such a fixing the currents $\{T, \overline{T}\}$ do not show up in the Hamiltonian (4.6) (they enter it only through M_0) and the system looks like a proper extension of the "ordinary" bosonic particle interacting with a constant magnetic field. However, the supercharges (4.5) still depend on $\{T, \overline{T}\}$ and cannot be constructed without such terms.

5 Some steps aside

Let us consider some relevant questions which are slightly aside our main direction in the paper.

5.1 Sphere

The most evident question is what is the situation with a particle moving over S^2 ? Whether we could introduce the interaction with a constant magnetic field in such a system too, preserving $\mathcal{N}=4$ supersymmetry? Leaving aside the superspace formulation, one may check that the supercharges

$$\begin{aligned} \mathcal{Q}_i &= (1 + x\bar{x})p\chi_i - i\bar{x}\hat{J}\chi_i + i\overline{\hat{T}}\bar{\chi}_i + \frac{i}{2}\bar{x}\chi^k\chi_k\bar{\chi}_i, \\ \overline{\mathcal{Q}}^i &= (1 + x\bar{x})\bar{p}\bar{\chi}^i + ix\hat{J}\bar{\chi}^i + i\hat{T}\chi^i + \frac{i}{2}x\bar{\chi}_k\bar{\chi}^k\chi^i, \end{aligned} \quad (5.1)$$

and the Hamiltonian

$$\mathcal{H} = (1 + x\bar{x})^2 \left(p - \frac{i\bar{x}}{1 + x\bar{x}} (\hat{J} - \chi_i\bar{\chi}^i) \right) \left(\bar{p} + \frac{ix}{1 + x\bar{x}} (\hat{J} - \chi_i\bar{\chi}^i) \right) - 2\hat{J}\chi_i\bar{\chi}^i + (\chi_i\bar{\chi}^i)^2 + \hat{T}\overline{\hat{T}}, \quad (5.2)$$

form the $\mathcal{N}=4$ Poincaré superalgebra (4.7), provided the hatted currents $\{\hat{T}, \overline{\hat{T}}, \hat{J}\}$ span $su(1,1)$ algebra

$$\{\hat{T}, \overline{\hat{T}}\} = -2i\hat{J}, \quad \{\hat{J}, \hat{T}\} = i\hat{T}, \quad \{\hat{J}, \overline{\hat{T}}\} = -i\overline{\hat{T}}. \quad (5.3)$$

The Hamiltonian (5.2) and the supercharges (5.1) are commuting now with the generators of the $su(2)$ algebra:

$$\begin{aligned} \tilde{R} &= p + \bar{x}^2\bar{p} + i\bar{x}\chi_i\bar{\chi}^i - i\bar{x}\hat{J}, \quad \overline{\tilde{R}} = \bar{p} + x^2p - ix\chi_i\bar{\chi}^i + ix\hat{J}, \\ \tilde{U} &= i(xp - \bar{x}\bar{p}) - \chi_i\bar{\chi}^i + \hat{J}. \end{aligned} \quad (5.4)$$

Thus the action (5.2) is invariant under $SU(2)$ transformations in full analogy with the $\mathcal{N}=4$ supersymmetric particle in the Lobachevsky space.

5.2 View from higher dimensions

In our Hamiltonian (4.6) and supercharges (4.5) the isospin variables enter through the $su(2)$ currents $\{T, \bar{T}, J\}$ as it should be. All that we need, in order to have a closed system, is that they commute as in (4.4): the structure of these currents is irrelevant. This suggests another interpretation of our system. Indeed, let us introduce the additional complex bosonic field u and the corresponding momenta p_u with the standard brackets

$$\{u, p_u\} = 1, \quad \{\bar{u}, \bar{p}_u\} = 1. \quad (5.5)$$

It is clear that now we may realize the currents $\{T, \bar{T}, J\}$ as

$$T = p_u + \bar{u}^2 \bar{p}_u, \quad \bar{T} = \bar{p}_u + u^2 p_u, \quad J = i(up_u - \bar{u}\bar{p}_u). \quad (5.6)$$

One may substitute these currents back into the Hamiltonian (4.6) and supercharges (4.5) to have now a four-dimensional $\mathcal{N}=4$ supersymmetric mechanics. In principle, one may make one step further and realize the currents $\{T, \bar{T}, J\}$ in terms of the complex bosonic field u , the additional real field ϕ and their momenta. The resulting system will be five-dimensional. The detailed analysis of this system will be reported elsewhere.

6 Conclusion

In the present paper we coupled the $\mathcal{N}=4$ chiral supermultiplet with an auxiliary $\mathcal{N}=4$ fermionic supermultiplet $\{\Xi^i, \bar{\Xi}^i\}$ containing on-shell four physical fermions and four auxiliary bosons. These bosons play the role of isospin variables. We choose a very specific coupling, resulting in a component action which contains only time derivatives of the fermionic components presented in $\{\Xi^i, \bar{\Xi}^i\}$. This structure of the on-shell action gives the possibility to dualize these fermions into auxiliary ones, ending up with the proper action for matter fields (bosons and fermions) and isospin variables. This procedure was developed in [6]. The conditions selecting such a coupling are very strong and the resulting component action (2.13) describes the interaction of the chiral supermultiplet with a magnetic field constant on the pseudo-sphere $SU(1,1)/U(1)$. That is why we then specify the prepotential of our theory to get in the bosonic sector the action for the particle moving over the pseudo-sphere – Lobachevsky space. We provided also the Hamiltonian formulation of this system and show that the full symmetry group of our Hamiltonian is $SU(1,1) \times U(1)$. The currents forming the $su(1,1)$ algebra are modified, as compared to the bosonic case, by the fermionic and isospin terms, while the additional $u(1)$ current contains only isospin variables.

One of the most important features of our construction is the presence in the Hamiltonian and supercharges of all currents of the isospin group $SU(2)$ – $\{T, \bar{T}, J\}$. Despite the fact that the currents $\{T, \bar{T}\}$ enter the Hamiltonian only through the Casimir operator of the $SU(2)$ group, they cannot be dropped out even after fixing of total isospin of the system. The reason for this behavior is simple: these currents themselves enter into the supercharges. Thus, the $\mathcal{N}=4$ supersymmetry insists on the presence of these currents in the system. One should stress that these currents appear automatically from our basic superspace action. Finally, we also present the Hamiltonian and supercharges describing the motion of a particle over the sphere S^2 in the background of a constant magnetic field. In this case the additional isospin currents $\{\hat{T}, \bar{\hat{T}}, \hat{J}\}$ form the $su(1,1)$ algebra.

The isospin variables enter our Hamiltonian and supercharges through the $su(2)$ currents $\{T, \bar{T}, J\}$ as it should be. The structure of these currents is irrelevant for our construction – it is enough that they span the $su(2)$ algebra. This fact opens a way to build these currents from additional bosonic variables, thus it gives rise to four- and/or five-dimensional $\mathcal{N}=4$ supersymmetric mechanics. We leave the analysis of these systems for the future.

The most relevant applications of our result are related with the Quantum Hall effect on the Lobachevsky space and/or S^2 with the constant background magnetic field, which can be thought of as due to a magnetic monopole seated in the center of a (pseudo) sphere [13, 12]. Clearly, the system we constructed in this paper provides their $\mathcal{N}=4$ supersymmetric generalizations. The last step one has to do to analyze the additional effects of $\mathcal{N}=4$ supersymmetry in the Quantum Hall effect is to construct the full quantum version of our Hamiltonian and supercharges. We are planning to consider the quantum version of our system in more details elsewhere.

Finally we note that it is possible and interesting to extend our results to higher dimensional $\mathcal{N}=4$ mechanics describing, for example, the particle moving over the CP^n manifold in the background of the

magnetic field of some monopole [9]. Indeed, our analysis shows that the $\mathcal{N}=4$ supersymmetrizations we performed preserve the initial $SU(1,1)$ and/or $SU(2)$ symmetry of the bosonic core. Hopefully, the same will be true for the system constructed in [14]. We are planning to report our results on the symmetries of the corresponding $\mathcal{N}=4$ supersymmetric system in a forthcoming paper [15].

Acknowledgements

We are indebted to Armen Nersessian for valuable discussions.

S.K. and A.S. thank the INFN-Laboratori Nazionali di Frascati, where this work was completed, for warm hospitality.

This work was partly supported by RFBF grants 11-02-01335, 11-02-90445-Ukr, as well as by the ERC Advanced Grant no. 226455, “*Supersymmetry, Quantum Gravity and Gauge Fields*” (*SUPERFIELDS*).

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